Joe Dinius
HW1

MATH 527A
November 28, 2008

1. A fundamental axiom for the real number $\mathbb{R}$ is that every bounded subset of $\mathbb{R}$ has a least upper bound or *supremum*.

   In what follows, $\{x_n\}$ is a bounded sequence, that is, there is a bound $C < \infty$ s.t. $|x_n| \leq C \forall n$. Let $l = \sup_n x_n$.

   (a) Show that, $\forall \epsilon > 0$, $\exists$ index m s.t. $x_m > l - \epsilon$.

   Solution. By contradiction, assume $\exists \epsilon > 0$, $\forall m$ s.t. $x_m < l - \epsilon$. Then $l - \epsilon$ is an upper bound for $\{x_n\}$, and $l - \epsilon < l$. Thus l is not the least upper bound of $\{x_n\}$. This is a contradiction. Thus for all $\epsilon > 0$, there is an index m such that $x_m > l - \epsilon$.

   (b) Show that $l \leq C$.

   Solution. Consider $l > C$, and hopefully a contradiction comes up. Since C is a bound on $\{x_n\}$, $|x_n| \leq C \ \forall n$. However, $l > C$, C would be an upper bound for $\{x_n\}$ smaller than l, which means l would no longer be the least upper bound. Contradiction. Therefore, $l \leq C$.

   (c) Find, with proof, $\sup_n x_n^2$.

   Claim. $\sup_n x_n^2 = (\sup_n |x_n|)^2$.

   Proof. Let $\sup |x_n| = l$. Then $\forall n$, $0 \leq |x_n| \leq l \implies |x_n|^2 \leq l^2$ and $x_n^2 \leq l^2$. Thus l^2 is an upper bound to x_n^2 Now assume there exists $0 < s < l$ such that $\sup_n x_n^2 = s^2$. then $\forall n$, $x_n^2 \leq s^2$ and $|x_n| \leq s$, and s is an upper bound for $\{|x_n|\}$. But $s < l$ and we assumed $l = \sup_n |x_n|$. This is a contradiction and thus l^2 is the least upper bound of $|x_n^2|$ and $\sup_n x_n^2 = l^2$. □

   (d) If x_n is a monotone nondecreasing sequence, show that $\lim_{n\to\infty} x_n$ exists.

   Solution. Suppose that x_n is bounded monotone nondecreasing sequence. Call X the nonempty bounded set $\{x_n : n \in \mathbb{N}\}$. Since X is bounded, there must be a least upper bound, call it l.
   Claim that $\lim_{n\to\infty} x_n = l$. Given any $\epsilon > 0$, $l - \epsilon$ is not an upper bound for X. Therefore, $\exists N$ s.t. $x_N > l - \epsilon$. Also, since the sequence x_n is nondecreasing and $l = \sup X$, the following holds

   $$l - \epsilon < x_N < \lim_{n\to\infty} x_n \leq l \ \forall n > N.$$

   Therefore, x_n converges to l, which means $\lim_{n\to\infty} x_n = l$.

(e) Prove this proposition from class: $a_n \geq 0$ is a sequence of non-negative reals. The infinite series $\sum_{n=1}^{\infty} a_n$ converges iff there is a constant $C < \infty$ s.t. $\forall N$ we have $\sum_{n=1}^{\infty} a_n \leq C$.

Proof. Let $S_N = \sum_{n=1}^{N} a_n$. Since $a_n \geq 0$ for all n, it follows that $S_{N+1} \geq S_N$ for all N. If S_N converges, it follows that the sequence S_N is bounded, so that there is a constant C s.t. for all N we have $\sum_{n=1}^{N} a_n \leq C$.

Alternatively, if $S_N \leq C$ for all N, it follows that S_N is bounded above, so that $l = \sup_N S_N$ exists. By the previous part, for any $\epsilon > 0$, there is an index m such that $S_m > l - \epsilon$. Since l is an upper bound, and S_k is a non-decreasing sequence, it follows that for all $k \geq m$, we have $l - \epsilon < S_k \leq l$, which implies $S_k \to l$, showing that the infinite series converges. □

2. (X, d_X) and (Y, d_Y) are metric spaces.

(a) If $U \subseteq X$, show that (U, d_U) is a metric space where $d_U(u_1, u_2) = d_X(u_1, u_2)\ \forall u_1, u_2 \in U$.

Solution. Pick $u_1, u_2 \in U$. Since, $U \subseteq X$, $u_1, u_2 \in X$, non-degeneracy holds for (U, d_U), since (X, d_X) is a metric space. That is, $d_U(u_1, u_2) = d_X(u_1, u_2) \geq 0$, $d_U(u_1, u_2) = d_X(u_1, u_2) = 0 \implies u_1 = u_2$.

Symmetry follows in a similar fashion: Take same u_1, u_2 as above. We know that $d_X(u_1, u_2) = d_U(u_1, u_2) = d_X(u_2, u_1) = d_U(u_2, u_1)$ since (X, d_X) is a metric space. So symmetry holds.

Finally, we need the triangle inequality to hold true. Start by adding $u_3 \in U$ to the other two elements in U already presented. Since (X, d_X) is metric space

$$d_X(u_1, u_3) \leq d_X(u_1, u_2) + d_X(u_2, u_3)$$

Use the relationship that $d_U(u_i, u_j) = d_X(u_i, u_j)\ \forall u_i, u_j \in U$.

$$d_U(u_1, u_3) \leq d_U(u_1, u_2) + d_U(u_2, u_3)$$

All four properties of metric spaces hold for (U, d_U). Therefore, (U, d_U) is a metric space.

(b) Show that (X, ρ) is a metric space, where

$$\rho(x_1, x_2) = \frac{d_X(x_1, x_2)}{1 + d_X(x_1, x_2)}$$

Solution. Since $d_X(x_1, x_2) \geq 0\ \forall x_1, x_2 \in X$, $1 + d_X(x_1, x_2) \geq 1$. Therefore, $\rho_X(x_1, x_2) \geq 0$. We have $\rho_X(x_1, x_2) = 0 \implies d_X(x_1, x_2) = 0$. This implies that $x_1 = x_2$, since (X, d_X) is a metric space. So, non-degeneracy holds.

For symmetry, use

$$\begin{aligned} d_X(x_1, x_2) &= d_X(x_1, x_2) \\ \rho(x_1, x_2) &= \frac{d_X(x_1, x_2)}{1 + d_X(x_1, x_2)} \\ &= \frac{d_X(x_2, x_1)}{1 + d_X(x_2, x_1)} \\ &= \rho(x_2, x_1); \end{aligned}$$

so symmetry holds.

To prove the triangle inequality, I will first need to prove a claim that is not immediately obvious.

Claim. *For any two nonnegative real numbers x and y, the following inequality holds:*

$$\frac{x+y}{1+x+y} \leq \frac{x}{1+x} + \frac{y}{1+y}.$$

Proof. Assume x, y have are nonnegative.

$$\begin{aligned} \frac{x+y}{1+x+y} &= \frac{x}{1+x+y} + \frac{y}{1+x+y} \\ &\leq \frac{x}{1+x} + \frac{y}{1+y}. \end{aligned}$$

The claim is true. The claim can be extended to all x, y using absolute values, however I don't need that much strength. □

We know that the triangle inequality holds on (X, d_X)

$$\begin{aligned} \rho(x_1, x_3) + \rho(x_3, x_2) &= \frac{d(x_1, x_3)}{1 + d(x_1, x_3)} + \frac{d(x_3, x_2)}{1 + d(x_3, x_2)} \\ &\geq \frac{d(x_1, x_3) + d(x_3, x_2)}{1 + d(x_1, x_3) + d(x_3, x_2)} \\ &= \frac{1}{1 + \frac{1}{d(x_1, x_3) + d(x_3, x_2)}} \\ &\geq \frac{1}{1 + \frac{1}{d(x_1, x_2)}} \\ &= \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} \\ &= \rho(x_1, x_2). \end{aligned}$$

All four properties of metric spaces hold. Therefore, (U, ρ) is a metric space.

(c) Show that the Cartesian product $X \times Y$ is a metric space with the metric $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Solution. First look at non-degeneracy: $d(z_1, z_2) \geq 0$ and $d(z_1, z_2) \implies z1 = z2$. Since (X, d_X) and (Y, d_Y) are metric spaces we know that $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2) \geq 0 \ \forall x \in X, y \in Y$ and $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0 \implies x_1 = x_2, y_1 = y_2$. Therefore

$$
\begin{aligned}
d(z_1, z_2) &\geq 0 \\
d(z_1, z_2) &= 0 \implies z_1 = z_2
\end{aligned}
$$

which shows non-degeneracy.

For symmetry,

$$
\begin{aligned}
d(z_1, z_2) &= d_X(x_1, x_2) + d_Y(y_1, y_2) \\
&= d_X(x_2, x_1) + d_Y(y_2, y_1) \\
&= d(z_2, z_1).
\end{aligned}
$$

This proves that symmetry holds.

Finally, as before, we need to show that the triangle inequality holds. Introduce $x_3 \in X, y_3 \in Y$.

$$
\begin{aligned}
d(z_1, z_3) &= d_X(x_1, x_3) + d_Y(y_1, y_3) \\
&\leq \big(d_X(x_1, x_2) + d_X(x_2, x_3)\big) + \big(d_Y(y_1, y_2) + d_Y(y_2, y_3)\big) \\
&= \big(d_X(x_1, x_2) + d_Y(y_1, y_2)\big) + \big(d_X(x_2, x_3) + d_Y(y_2, y_3)\big) \\
&= d(z_1, z_2) + d(z_2, z_3).
\end{aligned}
$$

Therefore all of the properties of metric spaces holds for $(X \times Y, d)$.

3. $\mathbf{x} \in \mathbb{R}^n$.

(a) Show that

$$\lim_{p \to \infty} ||\mathbf{x}||_p = \max(|x_1|, |x_2|, \ldots, |x_n|).$$

Solution. Start with the definition of $|| \cdot ||_p$:

$$||\mathbf{x}||_p = \Big(\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}}. \tag{1}$$

Let $\alpha = \max(x_i)$, which we know since the sequence is finite, thus $0 < \frac{x_i}{\alpha} \leq 1$. Definition (1) can be rewritten as

$$
\begin{aligned}
\lim_{p \to \infty} ||\mathbf{x}||_p &= \lim_{p \to \infty} \Big(\alpha^p \sum_{i=1}^{n} (\frac{x_i}{\alpha})^p\Big)^{\frac{1}{p}} \\
&= |\alpha| \lim_{p \to \infty} \Big(\sum_{i=1}^{n} (\frac{x_i}{\alpha})^p\Big)^{\frac{1}{p}}.
\end{aligned}
$$

As $p \to \infty$, $\left(\frac{x_i}{\alpha}\right)^p$ goes to 0 when $x_i \neq \alpha$; the only term in the sum that matters is where $x_i = \alpha$. That is,

$$\begin{aligned} \lim_{p\to\infty} ||\mathbf{x}||_p &= |\alpha| \\ &= \max(|x_1|, |x_2|, \ldots, |x_n|). \end{aligned}$$

(b) Show that, $\forall p, q \geq 1$,

$$\frac{||\mathbf{x}||_p}{n} \leq ||\mathbf{x}||_q \leq ||\mathbf{x}||_p.$$

Solution. For $p, q \geq 1$ use equations from Flaschka,

$$\begin{aligned} ||x||_\infty &\leq ||x||_k \quad (*) \\ ||x||_k &\leq n^{1/k}||x||_\infty \quad (**) \end{aligned}$$

for $1 \leq k \leq \infty$ and $x \in \mathbb{R}^n$. Then we can bound $||x||_q$ from above with eqn (*) and $k = q$.

$$||x||_q \leq n^{1/q}||x||_\infty$$

Likewise from below using eqn (**) and combining the two results

$$||x||_\infty \leq ||x||_q \leq n^{1/q}||x||_\infty$$

Now we also can bound the $||x||_\infty$ above by eqn (*) with $k = p$, and from below we have that $||x||_\infty \geq n^{-1/p}||x||_p$. Then,

$$n^{-1/p}||x||_p \leq ||x||_q \leq n^{1/q}||x||_p.$$

Since $n \geq 1$, $n^{1/k} \leq n$ in this domain ($n \in \mathbb{N}$) for any $k \geq 1$. So conclude that

$$n^{1/q} \leq n \text{ and } n^{-1/p} \geq n^{-1}$$

Therefore we can bound $||x||_q$ yet again to find the simpler result

$$\frac{||\mathbf{x}||_p}{n} \leq ||\mathbf{x}||_q \leq ||\mathbf{x}||_p.$$

4. Complete the proof of the proposition from class by showing that if $(i) 1 \leq p < q \leq \infty$, then $l^p(\mathbb{R}, \mathbb{N}) \subset l^q(\mathbb{R}, \mathbb{N})$, and (ii) there is a sequence $\{a_n\} \in l^q(\mathbb{R}, \mathbb{N})$, but $\notin l^p(\mathbb{R}, \mathbb{N})$.

   (i)

   Proof. Need to show that whenever

$$\sum_{i=0}^{\infty} |x_i|^p < \infty \text{ for } 1 \leq p < \infty, \tag{2}$$

$\sum_{i=0}^{\infty} |x_i|^q < \infty$ for $q > p$. Consider $(I) q = \infty$ and $(II) q < \infty$.

(I)
A sequence $\mathbf{x} = \{x_0, x_1, \dots\}$ for which (2) converges is bounded. Therefore

$$l^p(\mathbb{R}, \mathbb{N}) \subset l^q(\mathbb{R}, \mathbb{N}).$$

(II)

$$\begin{aligned}
\sum_{i=0}^{\infty} |x_i|^q &= \sum_{i=0}^{\infty} |x_i|^{q-p} |x_i|^p \\
&\leq \sum_{i=1}^{\infty} \left(\sup_j |x_i|^{q-p} \right) |x_i|^p \\
&= \left(\sup_j |x_i|^{q-p} \right) \sum_{i=1}^{\infty} |x_i|^p \\
&< \infty
\end{aligned}$$

which is interpreted as

$$l^p(\mathbb{R}, \mathbb{N}) \subset l^q(\mathbb{R}, \mathbb{N}).$$

Therefore, if $1 \leq p < q \leq \infty$, then $l^p(\mathbb{R}, \mathbb{N}) \subset l^q(\mathbb{R}, \mathbb{N})$. $\square$

(ii)

Proof. We just need to find a counterexample.

Define $\{a_n\} = \left\{ \left(\frac{1}{n}\right)^{\frac{1}{p}} : n \in \mathbb{N} \right\}$. Since $p < q$, $1 = \frac{p}{p} < \frac{p}{q}$. So

$$\sum_{i=0}^{\infty} \left(\frac{1}{i+1} \right)^{\frac{1}{p} \cdot p} = \sum_{i=0}^{\infty} \left(\frac{1}{i+1} \right) = \infty.$$

$\{a_n\} \notin l^p(\mathbb{R}, \mathbb{N})$. However,

$$\sum_{i=0}^{\infty} \left(\frac{1}{i+1} \right)^{\frac{1}{p} \cdot q} = \sum_{i=0}^{\infty} \left(\frac{1}{i+1} \right)^{\frac{q}{p}} < \infty$$

by the integral test. Therefore, the desired counterexample has been found; there is a sequence $\{a_n\} \in l^q(\mathbb{R}, \mathbb{N})$, but $\notin l^p(\mathbb{R}, \mathbb{N})$. $\square$

5. $0 < \alpha < 1$ and $0 < \beta < 1$. Find all $1 \leq p \leq \infty$ s.t. the "two-dimensional" sequence

$$x_{i,j} = \frac{1}{i^\alpha + j^\beta}$$

$\in l^p(\mathbb{R}, \mathbb{N})$.

Solution. We will first obtain a sufficient condition, that is we seek values for p which guarantee that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{|i^\alpha + j^\beta|^p}$$

converges. To evaluate this sum, we will consider it over three regions that partition the first quadrant of $\mathbb{R}^2$.

(i) $i^\alpha = j^\beta$. In this region $j = i^{\alpha/\beta}$ and the sum is

$$\sum_{i=1}^{\infty}\frac{1}{|i^\alpha + (i^{\alpha/\beta})^\beta|^p} = \sum_{i=1}^{\infty}\frac{1}{|2i^\alpha|^p}.$$

(ii) $i^\alpha > j^\beta$. In this region $j < i^{\alpha/\beta}$ and the sum is

$$\sum_{i=1}^{\infty}\sum_{j=1}^{i^{\alpha/\beta}}\frac{1}{|i^\alpha + j^\beta|^p} \le \sum_{i=1}^{\infty}\frac{i^{\alpha/\beta}}{|2i^\alpha|^p}.$$

(iii) $i^\alpha < j^\beta$. In this region $i < j^{\beta/\alpha}$ and the sum is

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j^{\beta/\alpha}}\frac{1}{|i^\alpha + j^\beta|^p} \le \sum_{j=1}^{\infty}\frac{j^{\beta/\alpha}}{|2j^\beta|^p}.$$

Putting these regions together we find that

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{|i^\alpha + j^\beta|^p} &\le \frac{1}{2^p}\left(\sum_{i=1}^{\infty}\frac{1}{|i^\alpha|^p} + \sum_{i=1}^{\infty}\frac{i^{\alpha/\beta}}{|i^\alpha|^p} + \sum_{j=1}^{\infty}\frac{j^{\beta/\alpha}}{|j^\beta|^p}\right)\\ &= \frac{1}{2^p}\left(\sum_{i=1}^{\infty} i^{-\alpha p} + \sum_{i=1}^{\infty} i^{\alpha(\frac{1}{\beta}-p)} + \sum_{j=1}^{\infty} j^{\beta(\frac{1}{\alpha}-p)}\right).\end{aligned}$$

The sum will converge if the sums over each of the regions converges. Thus, $\alpha p > 1$ and $\beta p > 1$ by symmetry, $\alpha(p - \frac{1}{\beta}) > 1$ and $\beta(p - \frac{1}{\alpha}) > 1$. Therefore the sum will converge and the sequence will be in $l^p(\mathbb{R}, \mathbb{N})$ for $p > \frac{1}{\alpha} + \frac{1}{\beta}$.

We now need to show that this condition is also necessary. That is, we will show that, for $1 \le p \le \frac{1}{\alpha} + \frac{1}{\beta}$, the series $\sum\sum |x_{i,j}|^p$ diverges.

We again consider the three regions as above, and assume that $p \le \frac{1}{\alpha} + \frac{1}{\beta}$. The argument for (i) is immediate. In (ii), $j < i^{\alpha/\beta}$ and

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{i^{\alpha/\beta}}\frac{1}{|i^\alpha + j^\beta|^p} &\ge \sum_{i=1}^{N}\frac{i^{\alpha/\beta}}{|i^\alpha|^p}\\ &\ge \sum_{i=1}^{N}\frac{i^{\alpha/\beta}}{i^{1+\alpha/\beta}}\\ &= \sum_{i=1}^{N}\frac{1}{i}\end{aligned}$$

which diverges as $N \to \infty$. The same argument works for (iii). So this condition is necessary.

6. Let $0 < p < 1$.

   (a) Show that $d_p(x, y) = |x - y|^p$ defines a metric on $\mathbb{R}$.

   Solution. The properties of non-degeneracy and symmetry follow directly from the properties of $|\cdot|$ on $\mathbb{R}$

$$\begin{aligned} |x-y| &\geq 0 \ \forall x, y \implies |x-y|^p \geq 0 \\ |x-y| &= 0 \implies x = y \implies |x-y|^p = 0 \implies x = y \\ |x-y| &= |y-x| \implies |x-y|^p = |y-x|^p. \end{aligned}$$

   For the triangle inequality, first observe for concave functions $f : \mathbb{R} \to \mathbb{R}$ that

$$f(x+y) \leq f(x) + f(y) \quad \forall x, y \geq 0.$$

   Given the triangle equality on $\mathbb{R}$, we have

$$\begin{aligned} |x-z| = |x-y+y-z| &= |(x-y)+(y-z)| \\ &\leq |x-y| + |y-z|. \end{aligned}$$

   Consider $f(t) = t^p$, $0 < p < 1$, $t > 0$. Clearly f is concave. Using the triangle inequality on $\mathbb{R}$ and the property given above for concave functions (comes from MVT), we have

$$|x-z|^p \leq (|x-y| + |y-z|)^p \leq |x-y|^p + |y-z|^p.$$

   So the triangle inequality holds.

   Therefore, $(\mathbb{R}, d_p)$ is a metric space.

   (b) Show that,

$$d_p(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|^p$$

   defines a metric on $\mathbb{R}^n$.

   Solution. The same arguments as in part (a) work here since:

$$\begin{aligned} |x_i - y_i| &\geq 0 \implies \sum_{i=1}^{n} |x_i - y_i| \geq 0 \implies \sum_{i=1}^{n} |x_i - y_i|^p \geq 0 \\ \big(|x_i - y_i| &= 0 \implies x_i = y_i\big) \implies \left(\sum_{i=1}^{n} |x_i - y_i| = 0 \implies x_i = y_i \ \forall i\right) \\ &\implies \left(\sum_{i=1}^{n} |x_i - y_i|^p = 0 \implies x_i = y_i \ \forall i\right) \\ \big(|x_i - y_i| &= |y_i - x_i| \implies |x_i - y_i|^p = |y_i - x_i|^p\big) \implies \left(\sum_{i=1}^{n} |x_i - y_i|^p = \sum_{i=1}^{n} |y_i - x_i|^p\right). \end{aligned}$$

Therefore, $d_P(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^n$ is non-degenerate and symmetric. Now for the triangle inequality.

From (a),

$$|x - z|^p \leq |x - y|^p + |y - z|^p.$$

Using this relationship at each term in the sum leads to

$$\sum_{i=1}^{n} |x_i - z_i|^p \leq \sum_{i=1}^{n} |x_i - y_i|^p + \sum_{i=1}^{n} |y_i - z_i|^p.$$

Therefore, $(\mathbb{R}^n, d_p)$ is a metric space.

(c) Can this definition be generalized to sequence spaces? For each p, what is the appropriate set of sequences on which the metric is defined.

Solution. The appropriate space to which this can be generalized is all sequences such that

$$\sum_{k=0}^{\infty} |x_k|^p < \infty.$$